Vector bundles on $\overline{\mathcal{M}}_{g,n}$ from vertex algebras

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This talk is about joint work with

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Bundles of coinvariants

$$\begin{array}{ccc} \mathbb{V}_g(V;\mathbf{M}^\bullet) & \qquad & \mathbb{V}_g(V;\mathbf{M}^\bullet)_{(C,P_\bullet)} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & & (C,P_\bullet) \end{array}$$

fibers are vector spaces of coinvariants

$$\mathbb{V}_g(V;\mathbf{M}^\bullet)_{(C,P_\bullet)} := [\mathbf{M}^\bullet]_{\mathcal{L}_{(C,P_\bullet)}(V)}.$$

Vector spaces of coinvariants are quotients

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For

- $(C, P_\bullet)$ a stable n-pointed curve; and
- $M^1, \ldots, M^n$ finitely generated admissible modules over a vertex operator algebra V,

the vector space of coinvariants

$$[\mathrm{M}^\bullet]_{\mathcal{L}_{(C,P_\bullet)}(V)} = \mathrm{M}^\bullet / \mathcal{L}_{(C,P_\bullet)}(V) \cdot \mathrm{M}^\bullet$$

is the largest quotient of the tensor product

$$\mathrm{M}^\bullet = \otimes_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{M}^{\mathrm{i}}$$

by the action of a Lie algebra

$$\mathcal{L}_{(C,P_\bullet)}(V).$$

To define these quotients

will describe the Lie algebra $\mathcal{L}_{(C,P_\bullet)}(V)$, and how it acts on the tensor product of the modules $\mathbf{M}^\bullet = \otimes_{i=1}^n M^i$.

There are two Lie algebras that act.

Surprisingly, their coinvariants are isomorphic (DGT2).

Before defining vertex operator algebras, and the main ingredients that go into the construction of the bundles, I briefly state our main results.

Summary of results (DGT I, DGT 2, DGT 3)

I. Vector spaces of coinvariants form quasi-coherent sheaves on $\overline{\mathcal{M}}_{g,n}$. If the modules are simple, they support a projectively flat connection.

II. If V is (a) of CFT type, (b) rational, and (c) C_2-cofinite, then vector spaces of coinvariants

1. *are finite dimensional;*
2. *satisfy factorization; and*
3. *are fibers of a vector bundle $\mathbb{V}_g(V; \mathbf{M}^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$.*
4. *The Chern character gives rise to a semi-simple CohFT.*

A VOA that is

(a) of CFT type, (b) rational, and (c) C_2-cofinite,

is said to be of CohFT-type.

I'll define these terms later in the talk.

Results from representations of affine Lie algebras

(1987) Tsuchiya and Kanie: defined coinvariants for smooth $g = 0$ with coordinates.

(1991) Tsuchiya, Ueno, and Yamada: coordinatized $(C, P_\bullet) \in \overline{\mathcal{M}}_{g,n}$; sheaves satisfy I and II (1-3)

(1993) Tsuchimoto: the vector bundles and the connection descend to $\overline{\mathcal{M}}_{g,n}$.

(2010) Fakhruddin: glob. gen. and first Chern class $g = 0$.

(2015) Marian, Oprea, Pandharipande: first Chern class in the tautological ring.

(2017) Marian, Oprea, Pandharipande, Pixton, Zvonkine show Chern characters give CohFT, derive expression.

Zhu's coinvariants (generalizing (TUY))

(1994) Zhu defined coinvariants and conjectured factorization for quasi primary generated (qpg) V.

(2005) Abe and Nagatomo VS finite dim for C smooth, and qpg V satisfying (a) & (c).

(2005) Nagatomo and Tsuchiya (2005) show I and II (1-3) for $g = 0$, $V \cong V'$, different hypothesis, extending Zhu's coinvariants for $g = 0$ curves with singularities.

(2005) Huang proved factorization for $g \in \{0, 1\}$, $V \cong V'$ satisfying (a),(b) & (c).

(2019) Codogni showed factorization, all g, for holomorphic $V \cong V'$ satisfying (a),(b) & (c).

Virasoro & FBZ coinvariants

(1991) Beilinson, Feigin, and Mazur construct coinvariants and prove factorization for coordinatized curves and the Virasoro VOA.

(2004) Frenkel and BenZvi define sheaves of coinvariants for coordinatized points $(C, P_\bullet) \in \mathcal{M}_{g,n}$ and modules over conformal vertex algebras, generalizing (BFM). They show sheaves support a projectively flat connection and mention that factorization is expected if V is rational.

(2019) (DGT1), (DGT2), and (DGT3)

(2020) We're working on applications using Chern classes.

New Examples

Vertex algebras of CohFT-type include most in the literature, including:

- Positive definite even lattice VOAs V_L and related “lattice type VOAs" ;
- Holomorphic VOAs like the moonshine module $V^\natural$;
- Special examples like $V_\ell(\mathfrak{g})$, Virasoro, and generalizations.

New VOAs of CohFT-type from old ones:

1. If $V^1, \ldots, V^k$ of CohFT type then $V^1 \otimes \cdots \otimes V^k$ is too;
2. commutants/cosets; and
3. orbifold algebras.

Commutant and coset examples

Commutant or Coset
For U a vertex subalgebra of V, conjecturally, if U and V are both of CohFT-type, then $\mathrm{Com}_V(U)$ is also of CohFT-type.

Orbifold algebras Let $G \subset Aut(V)$. The *orbifold vertex algebra* V^G consists of the fixed points of G in V. If V is of CohFT-type, $G = Aut(V)$ is a finite-dimensional algebraic group. If G is also solvable, then V^G will also be of CohFT-type. Conjecturally, V^G is always of CohFT-type.

Website with examples/info about many CohFT VOAs:
`https://www.math.ksu.edu/~gerald/voas/`

Why study these vector bundles?

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Here are three reasons, based on what is known to be true for the Verlinde bundles:

1. They may provide new examples of rational conformal field theories (I don't have much to say about this).
2. Give rise to elements in the tautological ring, and may be useful in testing Pixton's conjectures.
3. May help study birational geometry of $\overline{\mathcal{M}}_{g,n}$.

Chern classes of the bundles $\mathbb{V}_g(V; \mathbf{M}^\bullet)$ lie in the tautological ring.

Question

Do they obey Pixton's relations?

Global generation

If $\mathbb{V}_g(V; M^\bullet)$ is globally generated
$\implies c_1(\mathbb{V}_g(V; M^\bullet))$ is base point free.

Question

Can one specify conditions so that the bundles $\mathbb{V}_g(V; M^\bullet)$ *are generated by their global sections?*

Brief incomplete definition (mainly for notation)

A VOA is a tuple $\left(V, \mathbf{1}^V, \omega, Y(\cdot, z)\right)$, where

- $V = \oplus_{i \in \mathbb{N}} V_i$ is a $\mathbb{C}$-vector space, with $dim V_i < \infty$;
- $\mathbf{1}^V \in V_0$ (the vacuum vector),
- $\omega \in V_2$ (the conformal vector);
- $Y(\cdot, z)\colon V \to End(V)[[z, z^{-1}]]$ is a linear function assigning to every element $A \in V$ the *vertex operator*

$$Y(A, z) := \sum_{i \in \mathbb{Z}} A_{(i)} z^{-i-1}.$$

The datum $\left(V, \mathbf{1}^V, \omega, Y(\cdot, z)\right)$, referred to as V, must satisfy a number of axioms (look these up if interested).

The conformal structure comes from coefficients of the vertex operators

$$Y(\omega, z) = \sum_{i \in \mathbb{Z}} \omega_{(i)} z^{-i-1}.$$

Endomorphisms $L_p := \omega_{(p+1)}$, are subject to the Virasoro relations, giving the action of a Virasoro Lie algebra on V

$$\left[\omega_{(p+1)}, \omega_{(q+1)}\right] = (p-q)\, \omega_{(p+q+1)} + \frac{c}{12}\, \delta_{p+q,0}\, (p^3 - p)\, \mathrm{id}_V.$$

Here $c \in \mathbb{C}$ is the *central charge* of V.

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$\omega_{(1)}|_{V_i} = i \cdot \mathrm{id}_V$, for all i (so L_0 acts like a grading operator).

and $Y\left(\omega_{(0)} A, z\right) = \partial_z Y(A, z)$ (so L_{-1} acts like a derivative).

V-modules (also incomplete)

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A pair $(M, Y^M(- , z))$ with vector space $M = \oplus_{d \in \mathbb{N}} M_d$, and vertex operators

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satisfying hypothesis, depending on type of module.

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If M is simple, then for $v \in M_d$.

$$L_0(v) = (d + a)v.$$

Here $a \in \mathbb{C}$ is the conformal dimension of M.

CohFT-type

A VOA of CohFT type if

(a) V is of CFT type

$V_0 \cong \mathbb{C}$;

(b) V is rational

finitely generated admissible modules
are completely reducible;

(c) V is C_2-cofinite

$$\dim(V/C_2(V)) < \infty,$$
$$C_2(V) = \mathrm{Span}_{\mathbb{C}}\{A_{(-2)}B : A, B \in V\}.$$

Vertex algebras of CohFT-type have good properties:

- If V is rational (or V is C_2-cofinite) there are just finitely many simple modules.
- If V is rational and C_2-cofinite, the simple admissible modules are the same as the simple ordinary modules (DLM, 1997 Remark 2.4).
- Ordinary modules satisfy additional finiteness conditions, including
  1. graded pieces M_λ are finite dimensional
  2. for fixed λ, one has $M_{\lambda+\ell} = 0$ for $\ell >> 0$.

$V_\ell(\mathfrak{g})$

Lattice VOAs

For L be a free abelian group of finite rank d together with a positive-definite bilinear form $(\cdot,\cdot)$ such that $(\alpha,\alpha) \in 2\mathbb{Z}$ for all $\alpha \in L$.

The associated *even lattice vertex algebra* V_L has finitely many simple modules $\{V_{L+\lambda} \mid \lambda \in L'/L\}$, where L' is the dual lattice, and contragredients are: $V'_{L+\lambda} = V_{L-\lambda}$.

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- V_L is of CohFT-type with central charge $c = d$;
- The conformal dimension of the module $V_{L+\lambda}$ is $\frac{(\lambda,\lambda)}{2}$;

We next define the Lie algebras and their actions on modules. To describe the actions we first introduce the “ancillary Lie algebra".

Ancillary Lie algebra

Given a pointed curve (C, P), and t a local parameter on C at P, let

$$\mathcal{L}_P(V) = V \otimes \mathbb{C}((t))/Im\nabla,$$

where $\nabla : V \otimes \mathbb{C}((t)) \longrightarrow V \otimes \mathbb{C}((t))$, is the map

$$A \otimes f \mapsto L_{-1}A \otimes f + A \otimes \frac{d}{dt}f.$$

Generators and relations

$\mathcal{L}_P(V)$ has generators

$$\overline{A \otimes t^j} := A_{[j]} \in \mathcal{L}_P(V) = V \otimes \mathbb{C}((t))/Im\nabla,$$

and relations

$$[A_{[j]}, B_{[k]}] = \sum_{\ell \geq 0} \binom{j}{\ell} (A_{(\ell)}(B))_{[j+k-\ell]}.$$

$\mathcal{L}_P(V)$ acts on $\otimes_i M^i$

For M^i a V-module "at $P_i \in C$"

$$\bigoplus_{i=1}^{n} \mathcal{L}_{P_i}(V) \times \otimes_i M^i \to \otimes_i M^i,$$

$$((\ldots, A_{[k_j]}, \ldots), (m_1 \otimes \ldots \otimes m_n)) \mapsto$$

$$\sum_{j=1}^{n} \cdots \otimes m_{j-1} \otimes A_{k_j}^{M^j}(m_j) \otimes m_{j+1} \otimes \cdots .$$

The vertex algebra bundle

To give an algebro-geometric view of $\mathcal{L}_P(V)$, we will use the vertex algebra bundle

$$\mathcal{V}_C \to C,$$

defined for a smooth curve C by Frenkel and BenZvi, and extended in (DGT1) to stable curves with singularities. The fiber over a a point $P \in C$ is (non-canonically) isomorphic to V (which is an infinite object, so this is non-standard).

Construction of $\mathcal{V}_C$ using conformal structure

Algebro-geometric view of $\mathcal{L}_P(V)$

Theorem (FBZ, DGT 1)

$$\mathcal{L}_P(V) \cong H^0(D_p^\times, \mathcal{V}_C \otimes \omega_C / Im\nabla),$$

where $D_p^\times$ is the punctured disc on C at p.

FBZ/Chiral Lie algebra

Given a stable pointed curve $(C, P_\bullet)$, set

$$\mathcal{L}_{(C,P_\bullet)}(V) := H^0(C \setminus \cup P_i, \mathcal{V}_C \otimes \omega_C / Im\nabla).$$

One can show that the restriction

$$H^0(C \setminus \cup P_i, \mathcal{V}_C \otimes \omega_C / Im\nabla) \longrightarrow \bigoplus_j H^0(D^\times_{P_j}, \mathcal{V}_C \otimes \omega_C / Im\nabla),$$

$$\sigma \mapsto (\sigma|_{D^\times_{P_1}}, \ldots, \sigma|_{D^\times_{P_n}})$$

is a map of Lie algebras.

Diagonal action by restriction.

$$\mathcal{L}_{(C,P_\bullet)}(V) \times \otimes_i M^i \to \otimes_i M^i,$$

defined by

$$(\sigma, m_1 \otimes \cdots \otimes m_n) \mapsto \sum_{j=1}^{n} \cdots \otimes m_{j-1} \ \otimes \sigma|_{D^{\times}_{P_j}} \cdot m_j \otimes \cdots .$$

The vector space of coinvariants

$$\mathbb{V}_{\mathfrak{g}}(V; \mathbf{M}^\bullet)_{(C,P_\bullet)} := [M^\bullet]_{\mathcal{L}_{(C,P_\bullet)}(V)}.$$

is the largest quotient of the tensor product $\otimes_i M^i$ on which $\mathcal{L}_{(C,P_\bullet)}(V)$ acts trivially.

Factorization

Factorization

Factorization enables one to transform fibers of the bundle to fibers defined on simpler curves. This leads to recursions, and allows one to make inductive arguments. This is the crucial ingredient allowing ranks of such bundles, and their Chern characters to be given explicitly.

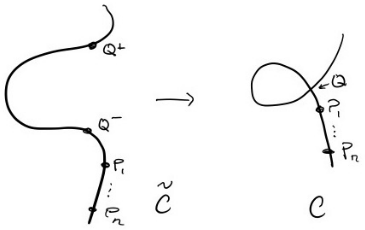
At curve with nonseparating node:

Theorem (Factorization)

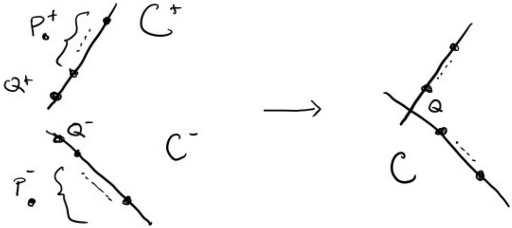
For V a conformal vertex algebra of CohFT-type, then with the notation as in the picture

$$\mathbb{V}(V; M^\bullet)_{(C,P_\bullet)} \cong \bigoplus_{W \in \mathcal{W}} \mathbb{V}\left(V; M^\bullet \otimes W \otimes W'\right)_{\left(\widetilde{C}, P_\bullet \sqcup Q_\bullet\right)},$$

where $\mathcal{W}$ is set of simple V-modules, and $Q_\bullet = (Q^+, Q^-)$.



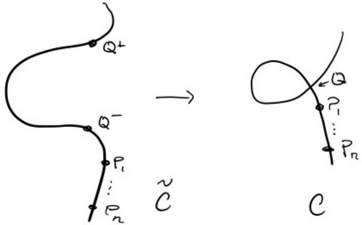
At a curve with a separating node:



$$
\begin{aligned}
\mathbb{V}\left(V; M^\bullet \otimes W \otimes W'\right)_{(\widetilde{C}, P_\bullet \sqcup Q_\bullet)} \\
\cong \bigoplus_{W \in \mathcal{W}} \mathbb{V}\left(V; M_+^\bullet \otimes W\right)_{X^+} \otimes \mathbb{V}\left(V; M_-^\bullet \otimes W'\right)_{X^-}
\end{aligned}
$$

where $X^\pm = (C^\pm, P_\bullet^\pm \sqcup Q^\pm)$, and $M_\pm^\bullet$ are the modules at the $P_\bullet^\pm$ on $C^\pm$.

The idea of the proof of factorization



Insert a trivial module Z at the two points of $\widetilde{C}$ lying over Q so coinvariants remain the same (trivial modules don't effect coinvariants).

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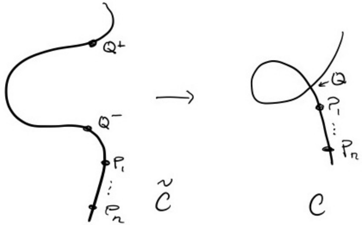

Insert a trivial module Z at the two points of $\widetilde{C}$ lying over Q so coinvariants remain the same (trivial modules don't effect coinvariants). This almost works.

We obtain a diagram

$$\begin{array}{ccc} [M^\bullet \otimes Z]_{\mathcal{L}_{(\widetilde{C}, P_\bullet \cup Q_\bullet)}(V)} & \xleftarrow{h} & [M^\bullet]_{\mathcal{L}_{(\widetilde{C}, P_\bullet)}(V,D)} \\ \downarrow & & \downarrow \\ [M^\bullet \otimes \overline{Z}]_{\mathcal{L}_{(\widetilde{C}, P_\bullet \cup Q_\bullet)}} & \xrightarrow{\cong} & [M^\bullet]_{\mathcal{L}_{(C, P_\bullet)}} \end{array} .$$

Finite dimensionality of the fibers is important to our argument. Taking duals, we work with vector spaces of conformal blocks via correlation functions.

Chern characters form a CohFT (DGT3)

Before describing a semisimple cohomological field theory (& other terms), we mention some consequences of this result extending (MOPPZ 2017).

Ranks

The ranks of $\mathbb{V}_g(V;\mathbf{M}^\bullet)$, which are the degree zero Chern classes of the bundles, form a topological quantum field theory (TQFT), and can be computed recursively from 3 point ranks (fusion rules).

Examples (DGT3)

Any bundle produced from a holomorphic vertex algebra of CohFT-type has rank one.

Let L be a positive definite even lattice such that $L'/L \cong \mathbb{Z}/m\mathbb{Z}$, for $m \geq 2$, and $\mathcal{W} = \{V = W_0, W_1, \ldots, W_{m-1}\}$ be the simple V_L-modules.

Then

$$\operatorname{rank} \mathbb{V}_g \left(V_L; W_0^{\otimes n_0} \otimes \cdots \otimes W_{m-1}^{\otimes n_{m-1}} \right) = m^g \, \delta_{\sum_{j=0}^{m-1} j n_j \equiv_m 0}.$$

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In particular, if $g = 0$ these also have rank one.

Corollary (DGT3)

1. *From the Chern character one can explicitly solve for the Chern classes; and*
2. *since the CohFT is semi-simple, these Chern classes lie in the tautological ring.*

First Chern class

For V a VOA of CohFT-type with central charge c, n simple V-modules M^i of conformal dimension a_i, $c_1\left(\mathbb{V}_g(V;M^\bullet)\right)$ is

$$\mathrm{\mathbf{rk}}\,\mathbb{V}_g(V;M^\bullet)\left(\frac{c}{2}\lambda+\sum_{i=1}^{n}a_i\psi_i\right)-b_{\mathrm{irr}}\delta_{\mathrm{irr}}-\sum_{i,I}b_{i:I}\delta_{i:I},$$

$$b_{\mathrm{irr}}=\sum_{W\in\mathcal{W}}a_W\cdot\mathrm{\mathbf{rk}}\,\mathbb{V}_{g-1}\left(V;M^\bullet\otimes W\otimes W'\right)$$

$$b_{i:I}=\sum_{W\in\mathcal{W}}a_W\cdot\mathrm{\mathbf{rk}}\,\mathbb{V}_i\left(V;M^I\otimes W\right)\cdot\mathrm{\mathbf{rk}}\,\mathbb{V}_{g-i}\left(V;M^{I^c}\otimes W'\right).$$

Simplest example:

V any holomorphic VOA of central charge c:

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These are all trivial on $\overline{\mathcal{M}}_{0,n}$.

Simple family from lattices, notation

Fix $m \geq 2$, and n integers $m_i \leq m-1$ such that $\sum_{1\leq i\leq n} m_i = km$, for some $k \geq 1$, and let (L, q) be an even lattice of (arbitrary) rank d for which $L'/L \cong \mathbb{Z}/m\mathbb{Z}$. From this data there is an associated vertex operator algebra V_L and n simple V_L-modules $W^i = V_{L+m_i}$ of conformal weight $c_i = \frac{q(m_i)}{2} \geq 0$ (q is the quadratic form).

We next give $c_1(\mathbb{V}_0(V_L; W^\bullet))$.

Simple family from lattices, first Chern class

With notation from prior slide, on $\overline{\mathcal{M}}_{0,n}$ one has

$$c_1\left(\mathbb{V}_0(V_L; W^\bullet)\right) = \sum_{i=1}^{n} c_i\psi_i - \sum_{I\subset[n], p_1\in I} \frac{q\Big(\sum_{i\in I} m_i\Big)}{2}\delta_{0,I},$$

where $q\Big(\sum_{i\in I} m_i\Big) = 0$ if $\sum_{i\in I} m_i \equiv 0 \ (mod\ m)$.

For $F_{A,B,C,D}$ any F-Curve on $\overline{\mathcal{M}}_{0,n}$, one has

$$\begin{aligned} & 2c_1\left(\mathbb{V}_0(V_L; W^\bullet)\right)\cdot F_{A,B,C,D} \\ &= q\Big(\sum_{a\in A} m_a\Big) + q\Big(\sum_{b\in B} m_b\Big) + q\Big(\sum_{c\in C} m_c\Big) + q\Big(\sum_{d\in A\cup B\cup C} m_d\Big) \\ &\quad - q\Big(\sum_{a\in A\cup B} m_a\Big) - q\Big(\sum_{a\in A\cup C} m_a\Big) - q\Big(\sum_{d\in B\cup C} m_d\Big). \quad (1) \end{aligned}$$

These are all nonnegative:
Lattice divisors like this are F-nef.

Question

What do these bundles tell us about $\overline{\mathcal{M}}_{g,n}$?

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This is what we are working on now.

The End (thank you).